

# Definiteness & optimization

CHEAT SHEET • 1 / 2 • Symmetric matrices and unconstrained optima

Assume  $A$  is a real symmetric  $n \times n$  matrix. For optimization,  $f$  is twice continuously differentiable and  $x^*$  is an interior stationary point:  $\nabla f(x^*) = 0$ . Let  $H = \nabla^2 f(x^*)$ .

## 1. Definitions and determinant tests

**Leading principal minor:**  $\Delta_k$  = determinant of the top-left  $k \times k$  block. **Principal minor:**  $\det A[I, I]$ , using the same index set  $I$  for rows and columns;  $k = |I|$ .

| Type                        | Quadratic form for every $v \neq 0$ | Equivalent determinant test                                                       |
|-----------------------------|-------------------------------------|-----------------------------------------------------------------------------------|
| Positive definite (PD)      | $v^T A v > 0$                       | $\Delta_k > 0$ for every $k$ .<br><b>Leading signs:</b> +, +, +, ...              |
| Negative definite (ND)      | $v^T A v < 0$                       | $(-1)^k \Delta_k > 0$ for every $k$ .<br><b>Leading signs:</b> -, +, -, ...       |
| Positive semidefinite (PSD) | $v^T A v \geq 0$                    | <b>All</b> principal minors $\geq 0$ .                                            |
| Negative semidefinite (NSD) | $v^T A v \leq 0$                    | For <b>every</b> principal minor of order $k$ :<br>$(-1)^k \det A[I, I] \geq 0$ . |

**Sylvester's criterion** uses leading minors for strict definiteness. The semidefinite extension requires **all principal minors**. Eigenvalue equivalents: all  $> 0$  /  $< 0$  /  $\geq 0$  /  $\leq 0$ , respectively. Semidefinite includes definite.

## 2. The $3 \times 3$ case

Write  $A$  with rows  $(a, b, c)$ ,  $(b, d, e)$ ,  $(c, e, f)$ . Let  $D = \det A = adf + 2bce - ae^2 - dc^2 - fb^2$ .

| PD / ND: 3 checks                                                                                  | PSD / NSD: 7 checks                                                                                                                                                        |
|----------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>PD:</b> $a > 0$ ; $ad - b^2 > 0$ ; $D > 0$ .<br><b>ND:</b> $a < 0$ ; $ad - b^2 > 0$ ; $D < 0$ . | <b>PSD:</b> $a, d, f \geq 0$ ; $ad - b^2, af - c^2, df - e^2 \geq 0$ ; $D \geq 0$ .<br><b>NSD:</b> $a, d, f \leq 0$ ; $ad - b^2, af - c^2, df - e^2 \geq 0$ ; $D \leq 0$ . |

## 3. What does the Hessian imply at a stationary point?

| Hessian $H$ at $x^*$                                | Conclusion                                                                                                                    |
|-----------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------|
| PD / ND                                             | PD $\Rightarrow$ strict local <b>minimum</b> .<br>ND $\Rightarrow$ strict local <b>maximum</b> .                              |
| PSD / NSD, with zero eigenvalues                    | <b>Inconclusive.</b> PSD is necessary for a local minimum; NSD is necessary for a local maximum. Neither alone is sufficient. |
| Indefinite (both positive and negative eigenvalues) | <b>Saddle point.</b>                                                                                                          |

**Why inconclusive?** At 0,  $f(x) = x^4$ ,  $-x^4$  and  $x^3$  all have  $f' = f'' = 0$ , yet give a minimum, maximum and neither, respectively. Strict definiteness is sufficient, but not necessary.

**Global distinction:**  $H(x)$  PSD everywhere on a convex domain implies convexity; NSD everywhere implies concavity. A stationary point is then a global minimum / maximum. A test at one point only gives local information.

# Constrained optimization

CHEAT SHEET • 2 / 2 • Equality constraints and the bordered Hessian

Optimize  $f(x)$  subject to  $g(x)=0$ , with  $m$  independent equality constraints and  $m < n$ . Assume  $f$  and  $g$  are  $C^2$  and  $J = Dg(x^*)$  has rank  $m$ . All matrices below are evaluated at a feasible Lagrange stationary point.

## 4. First-order conditions and feasible curvature

**Use one consistent convention:**  $L(x,\lambda) = f(x) - \lambda^T g(x)$ . Solve  $\nabla_x L = 0$  and  $g(x)=0$ . Let  $W = \nabla_{xx}^2 L$ . For nonlinear constraints,  $W$  includes their second derivatives.

Only tangent directions  $Jv = 0$  matter. Choose an  $n \times (n-m)$  matrix  $Z$  whose columns form a basis of this nullspace. The **reduced Hessian** is  $R = Z^T W Z$ .

| Curvature on tangent directions | Constrained conclusion                                                           |
|---------------------------------|----------------------------------------------------------------------------------|
| R PD / R ND                     | Strict local minimum / strict local maximum.                                     |
| R PSD / R NSD                   | Necessary for a local minimum / maximum.<br>With zero eigenvalues: inconclusive. |
| R indefinite                    | Neither a local minimum nor a local maximum.                                     |

## 5. Bordered Hessian: strict determinant tests

Form the symmetric block matrix  $B = \begin{bmatrix} 0_{m \times m} & J^T \\ J & W \end{bmatrix}$ , of size  $(m+n) \times (m+n)$ . The constraint rows and columns come first.

**B itself is indefinite** under these rank assumptions. We test definiteness of  $W$  **on the tangent space**, equivalently of  $R$ , using selected determinants of  $B$ .

**Variable ordering:** reorder variables so the first  $m$  columns of  $J$  form a nonsingular  $m \times m$  block. Let  $D_s$  be the determinant of the top-left  $s \times s$  block of  $B$ .

| Check only $s = 2m+1, \dots, m+n$ | Signs required for every $r = 1, \dots, n-m$                                 |
|-----------------------------------|------------------------------------------------------------------------------|
| Strict local <b>minimum</b>       | $(-1)^m D_{2m+r} > 0$ .<br>All checked determinants have sign $(-1)^m$ .     |
| Strict local <b>maximum</b>       | $(-1)^{m+r} D_{2m+r} > 0$ .<br>Alternate signs; first sign is $(-1)^{m+1}$ . |

These are sufficient conditions for strict optima, equivalent to  $R$  being PD / ND. Ignore the first  $2m$  leading minors when applying this test.

## 6. Most common case: one equality constraint ( $m = 1$ )

Ordering assumption:  $\partial g / \partial x_1 \neq 0$  at the candidate point (reorder variables if needed).

| Number of variables           | Strict minimum                      | Strict maximum                        |
|-------------------------------|-------------------------------------|---------------------------------------|
| $n = 2$ : $B$ is $3 \times 3$ | $\det B = D_3 < 0$                  | $\det B = D_3 > 0$                    |
| $n = 3$ : $B$ is $4 \times 4$ | $D_3 < 0$ and $D_4 < 0$             | $D_3 > 0$ and $D_4 < 0$               |
| General $n$                   | $D_3, D_4, D_5, \dots$ all negative | $D_3, D_4, D_5, \dots$ : +, -, +, ... |

**Zeros in the bordered test?** Do not simply weaken the inequalities. Compute  $R$  and apply the all-principal-minors semidefinite tests on page 1. Even  $R$  PSD / NSD alone does not establish an optimum.

**Inequality constraints:** use KKT conditions and curvature on the critical cone. The equality-constrained bordered test does not automatically apply to all active inequalities.

Reference for bordered-minor sign convention: [Hessian matrix - Bordered Hessian](#).